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THE STABILITY OF THE EQUILIBRIUM OF HOLONOMIC CONSERVATIVE SYSTEMS†

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The problem of the stability of the equilibrium positions and steady motions of holonomic conservative systems has been fairly completely treated in a number of reviews [44, 58, 9]. However, investigations are continuing in this field and a number of new important results have recently been obtained (in 1982–1992). This review analyses these results and compares them with previous ones.

BY THE well-known Lagrange–Dirichlet theorem [48, 49], the equilibrium position of a real system is stable if the potential has a strictly local minimum at this position. Although, as was shown by Painlevé [53] and, subsequently, by Wintner [67], this theorem is not invertible, it is nevertheless of interest to investigate the additional constraints, apart from the absence of a potential energy minimum, which lead to instability of an equilibrium. As is traditional, we shall subsequently refer to this problem as the problem of the inversion of the Lagrange–Dirichlet theorem used in its conventional sense. A fairly comprehensive review of this topic can be found in [9] (also, see [25, 21]). We shall therefore consider investigations which have been carried out more recently and are not mentioned in [9].

In all of the diverse approaches to the problem of inversion reflected in these investigations, it is possible nevertheless to pick out the most important trends within the framework of which instability theorems are obtained. Among these approaches are: (1) the first Lyapunov method, (2) the second Lyapunov method, (3) the variational method, and (4) the investigation of instability using the Hamiltonian action function

It is also noteworthy that the results obtained within the framework of the first Lyapunov method are grouped around the paper [12] while those relating to the second Lyapunov method, which is essentially based on the Chetayev instability theorem [41, p. 25] and also indirectly make use of Chetayev's theorem [39–41], are concerned with the inversion of the Lagrange–Dirichlet theorem.

Notation is subsequently introduced which is not always identical to that of other authors. This, however, has no effect on the content of the results described.

1. INVERSION OF THE LAGRANGE-DIRICHLET THEOREM USING THE FIRST LYAPUNOV METHOD

Consider a natural system with n degrees of freedom

$$d/dt\partial L/\partial \dot{\mathbf{q}} - \partial L/\partial \mathbf{q} = \mathbf{0} \tag{1.1}$$

where the Lagrangian L is defined by the expression

$$L(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - \Pi(\mathbf{q}) \tag{1.2}$$

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The function $T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T A(\mathbf{q})\dot{\mathbf{q}}$ corresponds to the kinetic energy of the system, and it is assumed that the quadratic form $\dot{\mathbf{q}}^T A(\mathbf{0})\dot{\mathbf{q}}$ is positive definite; $\Pi(\mathbf{q})$ is the potential energy. We shall assume that the point $\mathbf{q} = \dot{\mathbf{q}} = 0$ is the position of equilibrium being investigated.

Let us initially suppose that the functions $T(\mathbf{q}, \dot{\mathbf{q}})$ and $\Pi(\mathbf{q})$ are analytic in the neighbourhood of the equilibrium position in question and, in particular, that

$$\Pi(\mathbf{q}) = \Pi_m(\mathbf{q}) + \Pi_{m+1}(\mathbf{q}) + \ldots, \quad m \geqslant 2 \tag{1.3}$$

where $\Pi_s(\mathbf{q})$ is a homogeneous function of degree s: $\Pi_s(\lambda\mathbf{q}) = \lambda^s\Pi_s(\mathbf{q})$.

Without loss of generality, it may next be proposed that the normal coordinates, in which

$$A(\mathbf{q}) = E + A^*(\mathbf{q}), \quad A^*(\mathrm{o}) = 0$$

where E is the unit matrix, are chosen as the generalized coordinates. When this is taken into account, the equations of motion (1.1) can be represented in the form

$$\ddot{\mathbf{q}} = -\partial\Pi_m / \partial\mathbf{q} + O(\|\dot{\mathbf{q}}\|^2) + o(\partial\Pi_m / \partial\mathbf{q}) \tag{1.4}$$

Here, $O(\|\dot{\mathbf{q}}\|^2)$ are terms which, as regards their absolute magnitude, do not exceed a constant multiplied by $\|\dot{\mathbf{q}}\|^2$ and $o(\partial\Pi_m/\partial\mathbf{q})$ are quantities of a higher order of smallness in the neighbourhood of zero than $\|\partial\Pi_m/\partial\mathbf{q}\|$.

On discarding the last two terms in Eqs (1.4), which is equivalent, respectively, to the retention of the terms of smallest size in the expressions for the kinetic and potential energies of the initial system, we arrive at the truncated equation

$$\ddot{\mathbf{q}} = -\partial\Pi_m / \partial\mathbf{q} \tag{1.5}$$

Lemma 1 [11]. If, for a certain $\mathbf{e} \in R^n$ $(\|\mathbf{e}\| = 1)$ $(-\partial\Pi_m/\partial\mathbf{q})|_{\mathbf{q}=\mathbf{e}} = \kappa\mathbf{e}$, $m \geqslant 3$, $\kappa > 0$, then the truncated system (1.5) has the asymptotic solution

$$\mathbf{q}(t) = \mathbf{a}/t^{2/(m-2)}, \quad \mathbf{a} \in R^n, \quad \mathbf{a} = \|\mathbf{a}\|\mathbf{e} \tag{1.6}$$

The condition of the lemma is therefore equivalent to the requirement that the force $(-\partial\Pi_m/\partial\mathbf{q})$ is central and repulsive along a ray which is defined by the vector $\mathbf{e}$. If, in particular, the form $\Pi_m(\mathbf{q})$ does not have a minimum at the point $\mathbf{q} = \mathbf{0}$, then $(-\partial\Pi_m/\partial\mathbf{q})|_{\mathbf{q}=\mathbf{e}} = \kappa\mathbf{e}$ for certain $\mathbf{e} \in R^n$ and $\kappa > 0$ and the condition of the lemma is thereby satisfied.

It is well known [19] that the essence of the first Lyapunov method lies in finding a general or particular solution of the equations of the perturbed motion of the system under investigation which enables one to conclude whether its zeroth solution is stable or not. Here, the required solution, as a rule, is sought in the form of a series. The solution of the linear approximation of the equations of the perturbed motion usually serves as a basis for the latter series.

In particular, if $m = 2$ in (1.3) and the form Π_2 does not have a minimum at the point $\mathbf{q} = \mathbf{0}$, then, as was shown by Lyapunov [19, p. 24], the initial system (1.1) admits of a solution of the form

$$\mathbf{q}(t) = \sum_{i=1}^{\infty} \mathbf{q}_i e^{i\lambda t}, \quad \lambda < 0, \quad \mathbf{q}_i \in R^n \tag{1.7}$$

where the first term corresponds to a particular solution of the truncated system (1.50).

This fact serves as a directing consideration in also proceeding in an analogous manner when $m > 2$ by treating a system, that is not necessarily linear, as a first approximation.

It was shown in [11] that, in the case when $m \geqslant 3$, system (1.1) does actually admit of a solution which is constructed using this principle.

Theorem 1 [11]. If the expansion of the potential energy in a MacLaurin series starts from terms of odd powers, then an asymptotic solution of system (1.1) exists, and is expressed by the converging series

$$m = 2k+1, \quad \mathbf{q}(t) = \sum_{i=1}^{\infty} \frac{\mathbf{q}_i}{t^{i\mu}}, \quad \mu = \frac{2}{2k-1}, \quad \mathbf{q}_i \in R^n \tag{1.8}$$

and, in particular, the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ is unstable.

The situation is somewhat more complex in the case of even m. According to [12], in this case the solution has the form

$$m = 2k, \quad k \geqslant 2, \quad \mathbf{q}(t) = \frac{1}{t^{\mu}} \sum_{\substack{i,j=0 \\ j < \mu i}}^{\infty} \frac{\mathbf{q}_{ij}(\ln t)^j}{t^{i\mu}}$$
$$\mu = \frac{1}{k-1}, \quad \mathbf{q}_{ij} \in R^n \tag{1.9}$$

where $\mathbf{q}_{ij}$ are polynomials of $\ln t$.

Theorem 2 [12]. Suppose the form Π_m $(m \geqslant 3)$ does not have a local minimum at the point $\mathbf{q} = \mathbf{0}$. Then, a solution of (1.1) exists which is determined by series (1.7)–(1.9) and tends asymptotically to the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ as $t \to \infty$.

Corollary. Under the conditions of Theorem 2, the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1) is unstable.

It is seen that Theorem 2 includes Theorem 1.

In order to find the coefficients of series (1.8) and (1.9), we use induction, descending with respect to i and descending with respect to j. The existence of a real motion of system (1.1), for which the formal series which have been found serve as an asymptotic representation, follows from the results in [15, 16] (also, see [13, p. 934] and from the very fact that a formal solution exists in the form of series (1.8) and (1.9). However, in [11, 12], not only is a procedure for constructing the formal solution presented but the convergence of series (1.8) and (1.9) is also proved. For this purpose, system (1.1) is transformed to a certain equivalent operator form which permits the use of a theorem on a fixed point.

This method of investigating the stability of an equilibrium [11, 12] can be extended to the more-general case when the MacLaurin series for the function $\Pi(\mathbf{q})$ has the form

$$\Pi(\mathbf{q}) = \Pi_2 + \Pi_m + \Pi_{m+1} + \ldots, \quad m \geqslant 3$$

where Π_2 is a non-negative quadratic form. Noting that the set of points from R^n in which the equality $\Pi_2 = 0$ holds, forms a k-dimensional plane π containing the point $\mathbf{q} = \mathbf{0}$ and assuming that $k > 0$, we denote by W_m the value of the form Π_m in the plane π which is also a homogeneous form of power m.

Theorem 3 [13]. If the function W_m does not have a local minimum at the point $\mathbf{q} = \mathbf{0}$, then system (1.1) possesses motions, asymptotic to the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$.

Under the conditions of Theorem 3 and unlike in the case when $\Pi_2 \equiv 0$, it is no longer necessary to talk of the convergence of series of the form of (1.8) and (1.9). However, formal series with an analogous structure also exist in this situation, representing asymptotic expansions of solutions which are attracted to the equilibrium position when $t \to \infty$.

Theorem 3 also remains true in the case when $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^{\infty}(D \times R^n)$ only if the initial function $\Pi(\mathbf{q})$ can be represented in the form

$$\Pi(\mathbf{q}) = \Pi_2(\mathbf{q}) + \Pi_m(\mathbf{q}) + o(\|\mathbf{q}\|^m)$$

The results [11–13] described above have been further developed in [51, 52, 63] in the direction of a relaxing of the requirements on the smoothness of the corresponding Lagrangian. This relaxation is achieved, in particular, on account of the fact that only the existence of asymptotic solutions is proved rather than an explicit construction of the asymptotic solutions in the form of convergent series.

Theorem 4 [51, 52]. Let $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^h(s_\varepsilon \times R^n)$, $h > 2$ and suppose a natural number m, $2 < m \leqslant h$ exists such that

$$\Pi = \Pi_m + W$$

where Π_m is a form of degree m and the function W has a higher order of smallness at the point $\mathbf{q} = \mathbf{0}$. Then, if the form Π_m does not have a minimum at the point $\mathbf{q} = \mathbf{0}$, a motion of the natural system exists which is defined for $t \in]0, \infty[$ and tends to the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ as $t \to \infty$.

The proof of Theorem 4, like that of Theorem 2, is based on the use of the truncated equations (1.5). In particular, a replacement of the variables

$$\mathbf{q} = z(t)(\mathbf{e} + \mathbf{Q})$$

is carried out, where $\mathbf{q} = z(t)\mathbf{e}$ ($\mathbf{e} = (1, 0, \ldots, 0)^T$) is the solution of the truncated system. Then, by passing to the new system of variables, including the independent variable, the initial system (1.1) is transformed to the form of an autonomic system of $(2n+1)$ first-order equations which has a stable two-dimensional manifold. The latter fact enables us to draw a conclusion concerning the existence of an asymptotic motion of the initial system to the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of the initial system.

The scheme for the proof of this theorem can also be successfully used in the case of a more-general structure of the potential of the forces. In particular, representing the vector of the generalized coordinates $\mathbf{q}$ in the form of a pair $\mathbf{q} = (\mathbf{u}, \mathbf{v})$, $\mathbf{u} = (u_1, \ldots, u_r)$, $\mathbf{v} = (\upsilon_1, \ldots, \upsilon_s)$, $r + s = n$ and denoting the neighbourhood of the points $\mathbf{u} = \mathbf{0}$ in R^r and $\mathbf{v} = \mathbf{0}$ in R^s by Ω_1 and Ω_2, let us assume that the Lagrangian L for $(\mathbf{u}, \mathbf{v}, \dot{\mathbf{u}}, \dot{\mathbf{v}}) \in \Omega_1 \times \Omega_2 \times R^n$ has the form

$$L = \tfrac{1}{2}[\dot{\mathbf{u}}^T g^{(1)}(\mathbf{u},\mathbf{v})\dot{\mathbf{u}} + \dot{\mathbf{u}}^T g^{(c)}(\mathbf{u},\mathbf{v})\dot{\mathbf{v}} + \dot{\mathbf{v}}^T g^{(2)}(\mathbf{u},\mathbf{v})\dot{\mathbf{v}}] - \Pi(\mathbf{u},\mathbf{v}) \tag{1.10}$$

Here, $g^{(1)}(\mathbf{u}, \mathbf{v})$, $g^{(c)}(\mathbf{u}, \mathbf{v})$, $g^{(2)}(\mathbf{u}, \mathbf{v})$ are respectively the $(r \times r)$, $(r \times s)$ and $(s \times s)$ matrices $\forall(\mathbf{u}, \mathbf{v}) \in \Omega_1 \times \Omega_2$ which satisfy the conditions

$$\overset{(1)}{g}_{\alpha,\beta}(\mathbf{0}, \mathbf{0}) = \delta_{\alpha,\beta}, \quad \overset{(c)}{g}_{\alpha,j}(\mathbf{0}, \mathbf{0}) = 0, \quad \overset{(2)}{g}_{i,j}(\mathbf{0}, \mathbf{0}) = \delta_{i,j}$$

and Π, in the neighbourhood of the point $(\mathbf{u}, \mathbf{v}) = (\mathbf{0}, \mathbf{0})$, has a quadratic part

$$\Pi_2 = \delta_{\alpha,\beta}\omega_\alpha^2 u_\alpha u_\beta, \quad \omega_\alpha \neq 0, \quad \alpha = 1, \ldots, r$$

Suppose an integer $k \geqslant 3$ exists such that

$$H_1\colon\ L \in C^h(\Omega_1 \times \Omega_2 \times R^n, R), \quad h \geqslant k + 3$$

$$H_2\colon\ \Pi(\mathbf{u}, \mathbf{v}) = {}^1\!/_2\mathbf{u}^T l(\mathbf{u}, \mathbf{v})\mathbf{u} + \Pi_m(\mathbf{u}, \mathbf{v}) + W(\mathbf{u}, \mathbf{v})$$

where $l(\mathbf{u}, \mathbf{v})$ is an $(r \times r)$ matrix, $l_{\alpha,\beta}(\mathbf{0}, \mathbf{0}) = \omega_\alpha^2\delta_{\alpha,\beta}$, Π_m is a form of degree m, $W(\mathbf{u}, \mathbf{v}) = O(\|(\mathbf{u}, \mathbf{v})\|^{m+1})$ and $\min\{\Pi_m(\mathbf{0}, \mathbf{v}), \|\mathbf{v}\| = 1\} < 0$. Additionally, at least one of two supplementary conditions holds: H_3 (the weak coupling condition)

$$\hat{G}_{\alpha,\beta}(\mathbf{v}) = \delta_{\alpha,\beta} + O(\|\mathbf{v}\|^{m(k)+2}), \quad l_{\alpha,\beta}(\mathbf{0}, \mathbf{v}) = \omega_\alpha^2\delta_{\alpha,\beta} + O(\|\mathbf{v}\|^{m(k)+1})$$

where

$$\hat{G}_{\alpha,\beta}(\mathbf{v}) = \overset{(1)}{g}_{\alpha,\beta}(\mathbf{0},\mathbf{v}) - \overset{(c)}{g}_{\alpha,j}(\mathbf{0},\mathbf{v})(g^{(2)}(\mathbf{0},\mathbf{v}))^{-1}_{j,\kappa}\, \overset{(c)}{g}_{\beta,\kappa}(\mathbf{0},\mathbf{v})$$

$$H_3'\colon\ L \in C^{k+m(k)+3}, \quad m(k) = [(k-3)/2]$$

Theorem 5 [50]. Let a system with the Lagrangian (1.10) satisfy assumptions H_1 and H_2 and, also, one of the conditions H_3 or H_3'. The corresponding system (1.1) then has a solution which tends asymptotically to the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ (and the origin is therefore unstable).

Under the conditions of Theorem 5, coordinates $(\mathbf{x}, \mathbf{y})$, $\mathbf{x} \in R^r$, $\mathbf{y} \in R^s$, $r + s = n$ exist such that the potential energy can be represented in the form

$$\Pi = {}^1\!/_2\mathbf{x}^T r^*(\mathbf{x}, \mathbf{y})\mathbf{x} + \Pi_m(\mathbf{y}) + W(\mathbf{y}) \tag{1.11}$$

where $r^*(\mathbf{0}, \mathbf{0}) = \operatorname{diag}(\omega_1^2, \ldots, \omega_r^2)$ $(\omega_\alpha \neq 0,\ \alpha = 1, \ldots, r)$, Π_m is a form of the mth degree which does not have a minimum at the point $\mathbf{y} = \mathbf{0}$ and W is a function of a higher order of smallness. The very possibility of representing the function Π in the form of (1.11) enables one to pick out a non-autonomic subsystem from the initial system. This subsystem relates the variables $\mathbf{y}$, and the scheme for the proof of Theorem 4 is applicable to it. The latter fact is decisive in drawing a conclusion concerning the validity of Theorem 5.

The paper by Taliaferro [63], in which the solution of the truncated equations (1.5) is also used, is close in its conceptual plan to [12]. In particular, the required solution is represented in the form of a sum of the solution of the truncated equations and a certain small addition. In the paper being considered, there is a certain analogy with the approach proposed in [51, 52] since the idea of the transformation of the initial essentially non-linear system to a quasilinear system is also implemented.

Theorem 6 [63]. Let the following conditions be satisfied:

1. $T(\mathbf{q}, \dot{\mathbf{q}}): R^n \times R^n \to R$ is a continuously doubly differentiable, positive definite quadratic form for any $\mathbf{q} \in R^n$;

2. $\Pi(\mathbf{q})$, $\Pi_m(\mathbf{q}): R^n \setminus \{0\} \to R$ are double and triply continuously differentiable functions respectively, Π_m is a positively homogeneous function of degree m (that is, $\Pi_m(s\mathbf{q}) = s^m\Pi_m(\mathbf{q})$ for $s > 0$ and $\mathbf{q} \in R^n \setminus \{0\}$ and $\Pi^{(i)}(\mathbf{q}) = \Pi_m^{(i)}(\mathbf{q}) + O(\|\mathbf{q}\|^{m+\varepsilon-i})$ for $i = 0, 1, 2$ if $\|\mathbf{q}\| \to 0$);

3. $m > 1$ and $0 < \varepsilon \leqslant 1$ are real numbers and $n > 1$ is a natural number;

4. $\Pi_m(\mathbf{q}) < 0$ for a certain $\mathbf{q} \in R^n \setminus \{0\}$.

The equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1) is then unstable. If, however, $m \geqslant 2$, then a solution of the system exists which is asymptotically attracted to the point $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ as $t \to \infty$.

Remark. The original formulation of Theorem 6 is somewhat broader compared with that described above since it includes the case when $0 < m \leqslant 1$ and the points $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ is not the position of equilibrium which is outside the scope of the questions considered here.

Finally, in relation to the approach proposed in [11, 12], we recall a further paper [21] in which, using the procedure proposed in [12], solutions which are asymptotic to the equilibrium position are found in a situation with a more complex structure of the potential of the forces compared with that considered in [11–13].

As in [12], it is assumed that the initial Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ is analytic in the neighbourhood of the equilibrium position investigated and, in particular, that

$$\Pi(\mathbf{q}) = \Pi_{2m}(\mathbf{q}) + \Pi_{2m+2}(\mathbf{q}) + \ldots, \quad m \geqslant 3$$

Theorem 7 [21]. Let the following propositions be satisfied:

1. $\Pi_{2m}(\mathbf{q}) \geqslant 0$, $\forall \mathbf{q} \in R^n$, $S = \{\mathbf{q} : \Pi_{2m}(\mathbf{q}) = 0\}$;

2. $\exists \mathbf{e} \in \mathbf{S}$, $\|\mathbf{e}\| = 1$, $\Pi_{2m+2}(\mathbf{e}) = \min\{\Pi_{2m+2}(\mathbf{q}) : \|\mathbf{q}\| = 1\} < 0$;

3. $D^r\Pi_{2m}(\mathbf{e}) = 0$ when $r = 2, 3, 4$ ($D^r\Pi_k$ is a partial derivative of order r);

4. $\lim_{\|\mathbf{q}\| \to 0}[(A(\mathbf{q}) - A(\mathbf{0}))/\|\mathbf{q}\|^2] = 0$.

The equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1) is then unstable.

As has already been mentioned above, Theorem 7 is proved using the approach proposed in [12], according to which formal solutions of the form of (1.8), (1.9) are constructed as previously using induction ascending with respect to i and descending with respect to j for determining the coefficients of the series. Although, under the conditions of Theorem 7, the potential $\Pi(\mathbf{q})$ is more complex compared with that considered in [12] since it contains an additional term $\Pi_{2m}(\mathbf{q})$, this has no substantial effect on the structure of the operator equation which is equivalent to system (1.1) and, in particular, on the estimate of the norm of the operator. This enables us to draw a conclusion regarding the convergence of series (1.8) and (1.9).

The scheme for proving Theorem 7 with the additional constraints also enables us to encompass the more complex cases

1. $\Pi(\mathbf{q}) = \Pi_{2k}(\mathbf{q}) + \ldots + \Pi_{2m+1}(\mathbf{q}) + \Pi_{2m+2}(\mathbf{q}) + \ldots$;

2. $\Pi(\mathbf{q}) = \Pi_{2k}(\mathbf{q}) + \ldots + \Pi_{2m}(\mathbf{q}) + \Pi_{2m+1}(\mathbf{q}) + \ldots$ in which an assumption of the type (2) refers respectively to $\Pi_{2m+2}(\mathbf{q})$, $\Pi_{2m+1}(\mathbf{q})$.

Finally, in connection with the approach which is being considered here, on the basis of which the instability of an equilibrium is established by finding an asymptotic solution or proving the existence of an asymptotic solution, we also mention [3] in which (although there are more constraining assumptions) the existence of asymptotic solutions is also used to prove instability.

2. INVERSION OF THE LAGRANGE-DIRICHLET THEOREM USING THE SECOND LYAPUNOV METHOD

2.1. In the case of the results which were cited above [11–13, 21] and obtained using the first Lyapunov method, the requirement of analyticity or, at least, the infinite differentiability of the right-hand sides of the corresponding equations of the perturbed motion is essential. At the same time, if the second Lyapunov method, which also relies considerably on the lowest approximation of the equations of motion, is used as the basis for the subsequent analysis when investigating stability, then the constraints on the class of smoothness of the right-hand sides of the equations are reduced to a minimum. In particular, it is sufficient that the right-hand sides should ensure the existence and uniqueness of the solutions or even only their existence. Consequently, in those cases when the initial Lagrangian only satisfies the minimum smoothness requirements, the advantages of the second (direct) Lyapunov method for investigating stability are indisputable.

These facts were also precisely allowed for in [26, 32, 34] where the Lyapunov–Chetayev function is constructed for a wide class of conservative systems. The specific details of the use of the Chetayev instability theorem [41, p. 25] in [26, 32, 34] lies in the fact that the required properties of the auxiliary function and its derivative only refer to a subset of the zeroth level of the energy integral and not to the domain with a positive Lebesgue measure as is customarily assumed. This approach not only does not contradict Chetayev's theorem but, on the contrary, fairly fully implements the idea incorporated into it when, in order to draw a conclusion regarding instability, it is sufficient to detect just a single departing motion with an origin in as small a neighbourhood of the equilibrium position as may be desired. A similar constraint on the class of solutions considered simplifies the construction of the auxiliary function and the analysis of its derivative.

Theorem 8 [26]. Let $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^1(D \times R^n)$, $D \subset R^n_{\mathbf{q}}$ and, for a small number $\varepsilon > 0$ which may be as small as may be desired ($D \supset \bar{s}_\varepsilon = \{\mathbf{q} \in R^n, \ \|\mathbf{q}\| \leqslant \varepsilon\}$), let the following conditions be satisfied:

1. the potential energy $\Pi(\mathbf{q})$ can be represented in the form

$$\Pi(\mathbf{q}) = \Pi_k(\mathbf{q}) + R(\mathbf{q}), \quad R(\mathbf{q}) = o(\|\mathbf{q}\|^k)$$

where $\Pi_k(\mathbf{q})$ is a homogeneous form of degree $k \geqslant 2$;

2. the form $\Pi_k(\mathbf{q})$ does not have a minimum at the point $\mathbf{q} = \mathbf{0}$;

3. $\lim_{\|\mathbf{q}\| \to 0}(\|\partial R / \partial \mathbf{q}\| / \|\mathbf{q}\|^{k-1+\alpha}) = 0$, $\alpha = \text{const}$, $\alpha \in]0, 1[$.

The equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1) is then unstable.

The theorem is proved by representing system (1.1) in the Hamiltonian form

$$\dot{\mathbf{q}} = \partial H / \partial \mathbf{p}, \quad \dot{\mathbf{p}} = -\partial H / \partial \mathbf{q} \tag{2.1}$$

$$H(\mathbf{q}, \mathbf{p}) = {}^1\!/\!{}_2(\|\mathbf{p}\|^2 + \mathbf{p}^T B(\mathbf{q})\mathbf{p}) + \Pi(\mathbf{q}) = h = \text{const}$$

and considering the auxiliary function

$$V = \mathbf{q}\mathbf{p} - \delta\|\mathbf{q}\|^{k/2+1} e^{-\|\mathbf{q}\|^\alpha}, \quad 0 < \delta = \text{const}$$

in the set

$$\Lambda = \{(\mathbf{q}, \mathbf{p}) \in s^*_\varepsilon = \{(\mathbf{q}, \mathbf{p}) \in R^n \times R^n, \|\mathbf{q} \oplus \mathbf{p}\| < \varepsilon\}: \ H(\mathbf{q}, \mathbf{p}) = h = 0, \ V > 0\}$$

which is not empty with a suitable choice of the constant δ, regardless of the smallness of the number ε. In the case of the derivative of the function V along the vector field defined by Eqs (2.1), the conditions of the theorem guarantee the validity of the estimate

$$dV/dt > \alpha\delta^2\|\mathbf{q}\|^{k+\alpha} + o(\|\mathbf{q}\|^{k+\alpha}), \quad \forall(\mathbf{q}, \mathbf{p}) \in \Lambda$$

which enables one, by using the scheme for proving Chetayev's theorem, to draw a conclusion regarding the instability of the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$.

Remark 1. The structure of the set Λ enables us to obtain the estimate

$$\|\mathbf{q}(t)\| > \left[\|\mathbf{q}(0)\|^{-(k-2)/2} - \lambda\frac{(k-2)}{4}t\right]^{-2/(k-2)} \qquad k > 2$$

$$\|\mathbf{q}(t)\|^2 > \|\mathbf{q}(0)\|^2 e^{\lambda t}, \quad 0 < \lambda = \text{const}, \quad \lambda < 2\delta, \quad k = 2$$

for solutions which are departing from Λ and, thereby, from the neighbourhood s_ε^*.

Remark 2. The original formulation of Theorem 8 contains the further condition

$$\lim_{\|\mathbf{q}\|\to 0} \frac{A^*(\mathbf{q})}{\|\mathbf{q}\|^\alpha} = 0, \quad \alpha \in]0,1[$$

However, the latter is always satisfied as a consequence of the initial assumption that $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^1(D \times R^n)$.

This approach to the proof of instability can be extended to the more general case when the potential energy has the form

$$\Pi = P_m + R(\mathbf{q}), \quad P_m = \sum_{k=2}^{m} \Pi_k, \quad R(\mathbf{q}) = o(\|\mathbf{q}\|^m)$$

where Π_k is a homogeneous form of degree k.

Let us define the sets

$$\Omega = \{\mathbf{q} \in s_\varepsilon : \Pi(\mathbf{q}) < 0\}$$

$$\Omega_i = \left\{\mathbf{q} \in s_\varepsilon : P_i = \sum_{k=2}^{i \leq m} \Pi_k < 0\right\}, \quad \Omega_i^* = \Omega \cap \Omega_i$$

Theorem 9 [26]. Let $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^1(D \times R^n)$ and, for any $\varepsilon > 0$ $(D \supset \bar{s}_\varepsilon)$ as small as desired, let the following conditions be satisfied

1. $\Omega \neq \phi$;
2. $\Omega_m^* \neq \phi$;
3. $0 \in \overline{\Omega}_m^*$;
4. $\Omega_i^*\ (i < m) = \phi$;
5. $(-P_m) \geq \mu^2\|\mathbf{q}\|^m$, $\forall \mathbf{q} \in \omega \subset \Omega_m^*$, $\mu = \text{const}$, $0 \in \overline{\omega}$ (ω is a proper subset of the set Ω_m^*);
6. $\lim_{\|\mathbf{q}\|\to 0} (\|\partial R/\partial \mathbf{q}\|/\|\mathbf{q}\|^{m-1+\alpha}) = 0$, $\alpha \in]0,1[$.

The equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1) is then unstable.

Comparison of Theorem 9 with Theorem 5 shows that satisfaction of the conditions of the latter implies satisfaction of the conditions of Theorem 9 but the converse assertion does not hold. Consequently, Theorem 9 is an extension of Theorem 5.

Corollary. Let the potential energy $\Pi(\mathbf{q})$ in a sufficiently small neighbourhood of the equilibrium position, have the form

$$\Pi(\mathbf{q}) = \Pi_k(\mathbf{q}) + \Pi_m(\mathbf{q}) + R(\mathbf{q}), \quad R(\mathbf{q}) = o(\|\mathbf{q}\|^m)$$

where $\Pi_k(\mathbf{q})$, $\Pi_m(\mathbf{q})$ are, respectively, homogeneous forms of degree $k \geqslant 2$, $m > k$, $\Pi_k(\mathbf{q}) \geqslant 0$. The equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ is then unstable if the form $\Pi_m(\mathbf{q})$ in the set of zeros of the form $\Pi_k(\mathbf{q})$ can take negative values and the following condition is satisfied

$$\lim_{\|\mathbf{q}\| \to 0} (\|\partial R / \partial \mathbf{q}\| / \|\mathbf{q}\|^{m-1+\alpha}) = 0$$

A comparison of this corollary with the instability theorems 3 and 7 presented above shows that it generalizes the latter.

As in the case of Theorem 8, the proof of Theorem 9 is based on the use of the auxiliary function

$$V = \mathbf{qp} - \delta \|\mathbf{q}\|^{m/2+1} e^{-\|\mathbf{q}\|^\alpha}, \quad 0 < \delta = \text{const}$$

which is treated in a set of the type of Λ. The number m occurring in the expression for the auxiliary function reflects the fact that the polynomial P_m, like the form Π_k in the preceding case, is decisive when analysing the properties of the function V and its derivative relating to the set Λ.

The comparison of Theorem 9 with Chetayev's well-known result [40] is of interest. Here, the instability of the equilibrium position is proved subject to the conditions that $\Pi(\mathbf{q})$ is an analytic function and $\Omega_m^* \neq \phi$, $0 \in \overline{\Omega}_m^*$, $\Pi_k \geqslant 0$ $(k < m)$, $\Pi_k \leqslant 0$, $\forall\, k > m$. Although Chetayev's assertion regarding instability is more restricted compared with Theorem 9, the technique used to prove the latter nevertheless contains many of the elements of Chetayev's result.

As in [11–13], in the proof of Theorems 8 and 9 it is also important to make use of the possibility of separating the main part of the potential of the forces and, in particular, of the form Π_k and the polynomial P_m. However, the use of the auxiliary function V enables one to relax considerably the requirements concerning the smoothness of the corresponding Lagrangian in which the advantages of the second Lyapunov method are once again revealed.

The criterion for the instability of an equilibrium, which is also based on the method of Lyapunov functions and, in particular, rests on [22], has been obtained in [55, 56].

Theorem 10 [55, 56]. Let $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^2(s_\varepsilon \times R^n)$ and let the function $\Pi(\mathbf{q})$ not have a minimum at the point $\mathbf{q} = \mathbf{0}$. Then, if the following conditions are satisfied:

1. $\Pi(\mathbf{q}) = \Pi_m(\mathbf{q}) + R(\mathbf{q})$, where $\Pi_m(\mathbf{q})$ is a homogeneous form of degree $m \geqslant 2$,

2. $\dfrac{R(\mathbf{q})}{\|\mathbf{q}\|^m}, \dfrac{\partial R / \partial \mathbf{q}}{\|\mathbf{q}\|^{m-1}}, \dfrac{\partial^2 R / \partial \mathbf{q}^2}{\|\mathbf{q}\|^{m-2}} \to (0,0,0)$, if $\|\mathbf{q}\| \to 0$,

3. the function $\Pi_m(\mathbf{q})$ is non-degenerate $(\partial \Pi_m / \partial \mathbf{q} = \mathbf{0} \Rightarrow \mathbf{q} = \mathbf{0})$,

the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1) is unstable.

Theorem 10 somewhat strengthens the theorem in [22], in which it is assumed that $A(\mathbf{q}) = E$, $R(\mathbf{q}) \in C^m$ and $\partial R / \partial \mathbf{q}|_{\mathbf{q}=\mathbf{0}} = 0, \ldots, \partial^m R / \partial \mathbf{q}^m|_{\mathbf{q}=\mathbf{0}} = \mathbf{0}$. At the same time, the conditions in this theorem are more rigorous compared with those of Theorem 8.

As in [22], the function

$$V = \mathbf{p}^T(\mathbf{q} - \mu(\partial \Pi / \partial \mathbf{q})\|\mathbf{q}\|^{2-m}), \quad \mu = \text{const}$$

is used in the proof of Theorem 10.

The method of proving the instability of an equilibrium under the conditions of Theorems 8 and 9 enables one somewhat to broaden the domain of its application to more-complex structures of the force potential and, in particular, when the potential energy has the form

$$\Pi(\mathbf{q}) = \Pi_k(\mathbf{q}) + \Pi_m(\mathbf{q}) + R(\mathbf{q}), \quad R(\mathbf{q}) = o(\|\mathbf{q}\|^m)$$

Here, $\Pi_k(\mathbf{q})$, $\Pi_m(\mathbf{q})$ are homogeneous functions of degree $k>0$, $m>k$, respectively, $\Pi_k(\mathbf{q})\geq 0$, and not only homogeneous polynomials (forms) as was assumed above.

Theorem 11 [34]. Let $L(\mathbf{q}, \dot{\mathbf{q}})\in C^2(s_\varepsilon \times R^n)$ and, for a number $\varepsilon>0$ as small as desired $(D\supset \bar{s}_\varepsilon = \{\mathbf{q}\in R^n, \|\mathbf{q}\|\leq\varepsilon\})$ suppose the following conditions are satisfied:

1. the function W_m, by which the value of $\Pi_m(\mathbf{q})$ in the set of zeros of the function $\Pi_k(\mathbf{q})$: $\{\mathbf{q}\in R^n: \Pi_k=0\}$ is defined, does not have a minimum at the point $\mathbf{q}=\mathbf{0}$;
2. $\lim_{\|\mathbf{q}\|\to 0}(\|\partial R/\partial\mathbf{q}\|/\|\mathbf{q}\|^{m-1+\alpha})=0$, $\alpha=\mathrm{const}$, $\alpha\in]0, 1[$.

The equilibrium position $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ of system (1.1) is then unstable.

Corollary. Under the conditions of Theorem 11, solutions departing from the neighbourhood of the equilibrium position, exist and satisfy the estimates

$$\|\mathbf{q}(t)\|>\left[\|\mathbf{q}(0)\|^{-(m-2)/2}-\frac{\lambda(m-2)}{4}t\right]^{-2/(m-2)}, \quad m\neq 2$$

$$\|\mathbf{q}(t)\|^2>\|\mathbf{q}(0)\|^2e^{\lambda t}, \quad 0<\lambda=\mathrm{const}, \quad m=2$$

The scheme for the proof of Theorem 11 remains the same as in the case of Theorem 9. The fact that, in the situation under consideration, the homogeneous functions Π_k, Π_m do not necessarily reduce to forms is not of fundamental significance.

Theorem 12 [34]. Under the conditions of Theorem 11 and under the assumption that $L(\mathbf{q}, \dot{\mathbf{q}})\in C^2(D\times R^n)$, $m\geq 2$ asymptotic solutions exist which are attracted to the equilibrium position $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ as $t\to\infty$ and $t\to-\infty$ and which satisfy the inequalities

$$\|\mathbf{q}(t)\|<\left[\|\mathbf{q}(0)\|^{-(m-2)/2}\pm\lambda\frac{(m-2)}{4}t\right]^{-2/(m-2)}, \quad m>2$$

$$\|\mathbf{q}(t)\|^2<\|\mathbf{q}(0)\|^2e^{\mp\lambda t}, \quad m=2, \quad 0<\lambda=\mathrm{const}$$

where the upper and lower signs respectively refer to the values $t\in R^+=[0, \infty[$, $t\in R^-=]-\infty, 0]$.

The function $V=\mathbf{qp}+\sigma^*\exp(-\|\mathbf{q}^\alpha\|)\|\mathbf{q}\|^{m/2+1}$, $\sigma^*=\mathrm{const}$, which is considered in the set

$$\Lambda^*=\{(\mathbf{q}, \mathbf{p})\in \overset{*}{s}_\varepsilon: H(\mathbf{q}, \mathbf{p})=h=0, \ V<0\}$$

is used in the proof of Theorem 12.

The conditions of the theorem guarantee the validity of the estimate

$$dV/dt>\lambda\sigma^{*2}\|\mathbf{q}\|^{m+\alpha}+o(\|\mathbf{q}\|^{m+\alpha})\forall(\mathbf{q},\mathbf{p})\in\Lambda^*$$

and enable one to conclude that the set Λ^* is a sector.

In order to complete the theorem, it is sufficient to take account of the invertibility of the system under investigation as well as the possibility of representing the inequality occurring in the definition of Λ^* in the form

$$d/dt\|\mathbf{q}\|^2<-\lambda(\|\mathbf{q}\|^2)^{(m+2)/4}, \quad 0<\lambda<2\sigma^*$$

Theorem 12 is an extension of Theorem 3, in which a quadratic form corresponds to the homogeneous function $\Pi_k(\mathbf{q})$ and, moreover, $L\in C^{\omega(-)}$.

2.2. The work of Chetayev [40], associated with the construction of an auxiliary vector field possessing certain properties with respect to the potential of the forces of the system under investigation, occupies an important place in obtaining a number of new cases of the inversion of the Lagrange–Dirichlet theorem.

This result of Chetayev has found effective use in the investigations described in [47, 22, 10]. It has also been generalized to the case when the derivative of the auxiliary vector field undergoes a discontinuity. In spite of the large number of applications of Chetayev's work the role of the condition occurring in it of the absence of critical points of the function $\Pi(\mathbf{q})$ in the domain $\omega = \{\mathbf{q} \in s_\varepsilon : \Pi(\mathbf{q}) < 0\}$ has remained somewhat unclear. A partial answer to this question was given in [55, 56] where restrictions on the structure of the set of critical points of the function $\Pi(\mathbf{q})$ in ω were pointed out.

Theorem 13 [55, 56]. Let $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^2(D \times R^n)$ and, for a small number ε $(D \supset \bar{s}_\varepsilon)$ which may be as small as desired, let the set $\omega = \{\mathbf{q} \in s_\varepsilon : \Pi(\mathbf{q}) < 0\}$ be non-empty and $0 \in \partial\omega$. Let us suppose that a vector field

$$\varphi(\mathbf{q}) \in C^1 \colon\ s_\varepsilon \to R^n, \quad \varphi(\mathbf{0}) = \mathbf{0}$$

exists such that the following conditions are satisfied

1. $\varphi^T(\mathbf{q})\partial\Pi/\partial\mathbf{q} \leqslant 0, \ \forall \mathbf{q} \in \omega$;

2. $\mathbf{x}^T\left(\dfrac{\partial\varphi}{\partial\mathbf{q}}A(\mathbf{q})\right)\Bigg|_{\mathbf{q}=\mathbf{0}} \mathbf{x} \geqslant c\|\mathbf{x}\|^2, \quad \forall \mathbf{x} \in R^n, \quad 0 < c = \text{const}$;

3. $\forall \mathbf{q} \in \omega, (\forall \eta < 0)(\exists \eta' \colon\ \eta < \eta' < 0)\ (\Pi(\mathbf{q}) = \eta') \Rightarrow \partial\Pi/\partial\mathbf{q} \neq \mathbf{0}$.

The equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1) is then unstable.
The theorem is proved using the auxiliary function

$$V = (\partial L/\partial\dot{\mathbf{q}})^T(\varphi/\mathbf{q}) - \sigma\partial\Pi/\partial\mathbf{q}), \quad \sigma = \text{const}, \quad \sigma > 0$$

the derivative of which, by virtue of the equations of motion, satisfies the estimate

$$\dot{V} \geqslant \tfrac{1}{2}c\|\partial L/\partial\dot{\mathbf{q}}\|^2 + \sigma\|\partial\Pi/\partial\mathbf{q}\|^2$$

The condition of Theorem 3, which compensates for the relaxation of condition 1 compared with Chetayev's formulation, according to which $\varphi^T\partial\Pi/\partial\mathbf{q} < 0$ guarantees the choice of the motion with as small initial deviations from the equilibrium positions as desired for which $\dot{V} \geqslant \gamma = \text{const}$.
It turns out, however, that the following stronger theorem holds.

Theorem 14 [35]. Let $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^2(D \times R^n)$ and, for a number, which may be as small as desired, $\varepsilon > 0$ $(D \supset \bar{s}_\varepsilon)$, let $\omega \neq \phi$ and $0 \in \partial\omega$. Let us assume that a vector field $\varphi(\mathbf{q}) \in C^1 :\ s_\varepsilon \to R^n$ exists such that the following conditions are satisfied:

1. $\varphi^T(\mathbf{q})\partial\Pi/\partial\mathbf{q} \leqslant 0, \ \forall \mathbf{q} \in \omega$;

2. $\mathbf{x}^T\left(\dfrac{\partial\varphi}{\partial\mathbf{q}}A(\mathbf{q})\right)\Bigg|_{\mathbf{q}=\mathbf{0}} \mathbf{x} - \dfrac{1}{2}\varphi^T(\mathbf{0})\dfrac{\partial}{\partial\mathbf{q}}(\mathbf{x}^T A(\mathbf{q})\mathbf{x})|_{\mathbf{q}=\mathbf{0}} \geqslant c\|\mathbf{x}\|^2, \forall \mathbf{x} \in R^n, \ 0 < c = \text{const}$.

The equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1) is then unstable.
The function $V = \varphi^T(\mathbf{q})(\partial L/\partial\dot{\mathbf{q}})$ is used in the proof of this theorem and the derivative of the function in the motion of a system with the initial conditions

$$\varphi(\mathbf{q})|_{t=0} = \varphi(\mathbf{q}_0), \quad \partial L/\partial\dot{\mathbf{q}}\,|_{t=0} = \lambda\varphi(\mathbf{q}_0), \quad \|\varphi(\mathbf{q}_0)\| \neq 0$$

$$0 < \lambda = \text{const}, \quad T(\mathbf{q}_0, \dot{\mathbf{q}}_0) + \Pi(\mathbf{q}_0) = h_0 < 0.$$

satisfies the estimate $\dot{V} \geqslant c^* = \text{const}$. Since this choice of initial conditions does not imply any restrictions whatsoever on the structure of the set of critical points of the function $\Pi(\mathbf{q})$, the extension of the Theorem 13 is thereby also ensured.

Remark 1. The equality $\varphi(\mathbf{0})=\mathbf{0}$ is not assumed in Theorem 14, that is, the vector field $\varphi(\mathbf{q})$ may contain a constant component. If, however, we put $\varphi(\mathbf{0})=\mathbf{0}$, then condition 2 of Theorem 14 reduces to condition 2 of Theorem 13.

Remark 2. The assumption that $L(\mathbf{q},\ \dot{\mathbf{q}})\in C^2(D\times R^n)$, which guarantees the uniqueness of the solution, is not essential. Theorem 14 also remains true when $L(\mathbf{q},\ \dot{\mathbf{q}})\in C^1$. The requirement that $L(\mathbf{q},\ \dot{\mathbf{q}})\in C^2$, which usually occurs in the treatment of dynamical systems, can most probably be treated as a succession to the principle of Newtonian determinacy which is assumed in classical mechanics.

Theorem 15 [7]. Let the Lagrangian $L(\mathbf{q},\ \dot{\mathbf{q}})$ be analytic in the neighbourhood of the point $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ which is an isolated equilibrium position. Let us also assume that, at the point $\mathbf{q}=\mathbf{0}$, the potential energy $\Pi(\mathbf{q})=\Pi_m+\Pi_{m+1}+\ \ldots$ does not have a local minimum and that a vector field

$$\varphi(\mathbf{q})\in C^1\colon\ s_\varepsilon\to R^n \qquad \varphi(\mathbf{0})=\mathbf{0}$$

exists, and is such that

1. $\varphi^T(\mathbf{q})\partial\Pi/\partial\mathbf{q}\leqslant 0,\ \ \forall\mathbf{q}\in\omega$;

2. $\mathbf{x}^T\left(\dfrac{\partial\varphi}{\partial\mathbf{q}}A(\mathbf{q})\right)\Big|_{\mathbf{q}=\mathbf{0}}\mathbf{x}\geqslant c\|\mathbf{x}\|^2,\ \forall\mathbf{x}\in R^n,\ 0<c=\text{const.}$

The solution of system (1.1) with any initial position $\mathbf{q}(0)\in\omega$ and a sufficiently small momentum $\partial L/\partial\dot{\mathbf{q}}$ leaves the ε-neighbourhood of zero after a time T^*

$$c_1\|\mathbf{q}(0)\|^{(2-m)/2}\leqslant T^*\leqslant c_2\|\mathbf{q}(0)\|^{(m-2\alpha)/2}$$

where c_1, c_2 and α are constants. If the exponent $(2-m)/2$ or $(m-2\alpha)/2$ vanish (when $m=2$ or $m=2$, $\alpha=1$), then the corresponding power estimate is replaced by $\ln(\|\mathbf{q}(0)\|^{-1})$.

The theorem is proved using the scheme proposed in [22].

3. APPLICATION OF THE LYAPUNOV METHOD TO THE PROBLEM OF THE INVERSION OF ROUTH'S THEOREM

The methods for investigating the stability of an equilibrium which have been considered in Sections 1 and 2 can also be applied to the problem of the stability of steady motions [54, 59, 66]. It is well known [9, 24] that, in this case, the equations of the perturbed motion can be represented in the form of (1.1), where

$$L(\mathbf{q},\dot{\mathbf{q}})=L_2(\mathbf{q},\dot{\mathbf{q}})+L_1(\mathbf{q},\dot{\mathbf{q}})+L_0(\mathbf{q})={}^1\!/\!_2\dot{\mathbf{q}}^TA(\mathbf{q})\dot{\mathbf{q}}+\mathbf{f}^T(\mathbf{q})\dot{\mathbf{q}}+L_0(\mathbf{q}) \tag{3.1}$$

and, moreover, the quadratic form $\dot{\mathbf{q}}^TA(\mathbf{0})\dot{\mathbf{q}}$ is positive definite and the point $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ corresponds to the steady motion under investigation. Without loss of generality, it may be assumed that $\mathbf{f}(\mathbf{0})=\mathbf{0},\ L_0(\mathbf{0})=0$.

The criterion for the instability of steady motions—equilibria of system (1.1), (3.1) is usually known in the literature as the inversion of Routh's theorem. The latter [59] is a generalization of the Lagrange–Dirichlet theorem. The presence in the Lagrangian L of the term $L_1(\mathbf{q},\ \dot{\mathbf{q}})$ (only if $L_1(\mathbf{q},\ \dot{\mathbf{q}})\neq d\psi(\mathbf{q})/dt$) leads to the fact that the instability theorems which have been described above cannot be automatically transferred to the case of steady motions since gyroscopic stabilization [59, 41] can occur in system (1.1), (3.1).

Theorem 16 [26]. Let $L_1(\mathbf{q}, \dot{\mathbf{q}}) \in C^1(D \times R^n)$ and, for a number $\varepsilon > 0$ $(D \supset \bar{s}_\varepsilon)$, which may be as small as desired, let the following conditions be satisfied:

1. the function $L_0(\mathbf{q})$ can be represented in the form

$$L_0(\mathbf{q}) = P_m + R(\mathbf{q}), \quad P_m = \sum_{k=2}^{m} L_{0k}, \quad R(\mathbf{q}) = o(\|\mathbf{q}\|^m),$$

where L_{0k} are homogeneous functions of degree k;

2. $\Omega = \{\mathbf{q} \in s_\varepsilon\colon L_0(\mathbf{q}) > 0\} \neq \phi$;

3. $\overset{*}{\Omega}_i = \phi, \ \forall i < m, \overset{*}{\Omega}_i = \Omega \cap \Omega_i, \Omega_i = \{\mathbf{q} \in s_\varepsilon\colon P^i > 0\}$;

4. $\Omega_m^* \neq \phi, \ 0 \in \overline{\Omega}_m^*$;

5. $P_m \geqslant \mu^2 \|\mathbf{q}\|^m, \ \forall \mathbf{q} \in \omega \subset \overset{*}{\Omega}_m, \ \mu = \text{const}, \ 0 \in \bar{\omega}$,

where ω is a certain proper subset of the set Ω_m^*;

6. $\lim\limits_{\|\mathbf{q}\| \to 0} (\|\partial R / \partial \mathbf{q}\| / \|\mathbf{q}\|^{m-1+\alpha}) = 0, \quad \alpha = \text{const}, \quad \alpha \in]0,1[$ (3.2)

7. $\lim\limits_{\|\mathbf{q}\| \to 0} |\partial f_i / \partial q_j - \partial f_j / \partial q_i| / \|\mathbf{q}\|^{m/2-1+\alpha} = 0, \quad i,j = 1,2,\ldots,n.$ (3.3)

The steady motion (the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1), (3.1)) is then unstable.

Corollary. Let $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^1(D \times R^n)$ and, for a number $\varepsilon > 0$ which may be as small as desired, let the following conditions be satisfied:

1. the function $L_0(\mathbf{q})$ can be represented in the form

$$L_0(\mathbf{q}) = L_{0k}(\mathbf{q}) + R(\mathbf{q}), \quad R(\mathbf{q}) = o(\|\mathbf{q}\|^k),$$

where $L_{0k}(\mathbf{q})$ is a homogeneous form of degree $k \geqslant 2$;

2. the form $L_{0k}(\mathbf{q})$ does not have a maximum in the steady motion (at the point $\mathbf{q} = \mathbf{0}$);
3. equality (3.2) is satisfied.
4. equality (3.3) is satisfied.

The steady motion is then unstable.

Theorem 16 is proved according to the same scheme as in the case of Theorem 9. It is completely understandable that the possibility should be limited in this situation by the framework of the approach which is used to investigate the stability of the equilibrium of natural systems if it is taken into account that condition 7 of theorem reflects the fact that gyroscopic forces have a higher order of smallness as compared to potential forces in a set of the type of Λ defined in Section 2. Consequently, in accordance with condition 7 of the theorem, Eqs (1.1) with the Lagrangian (3.1) can be considered as a small perturbation of a natural system.

Theorem 17 [38]. Let the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ be analytic in a fairly small neighbourhood of the point $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$. If the first non-trivial form of the expansion of the function $L_0(\mathbf{q})$ in a MacLaurin series

$$L_0(\mathbf{q}) = L_{0k}(\mathbf{q}) + O(\|\mathbf{q}\|^{k+1}), \quad k \geqslant 2$$

does not have a maximum at the point $\mathbf{q} = \mathbf{0}$ and

$$\lim_{\|\mathbf{q}\| \to 0} (\|\mathbf{f}\|^2 / \|\mathbf{q}\|^k) = 0 \tag{3.4}$$

then the steady motion is unstable.

The proof of Theorem 17 is based on the use of the method in [12], according to which solutions of the system are constructed which are asymptotically attracted to the steady motion when $t \to \infty$ $(-\infty)$.

If, as a corollary to Theorem 16, we assume the analyticity of $L(\mathbf{q}, \dot{\mathbf{q}})$, we arrive at a criterion of instability which is equivalent to Theorem 17. In order to show that this is so, it is sufficient to note that condition (3) of the corollary, which is automatically satisfied by virtue of the analyticity of L, can be omitted while, in condition (4), the exponent $m/2-1$ is replaced by $k/2-1$ (see [32]).

It is clear that satisfaction of the modified equality (3.3) implies satisfaction of (3.4) and vice versa. Hence, also under the conditions of Theorem 17, although it was proved by another method, it is also actually assumed that the gyroscopic forces have a higher order of smallness compared with $\partial L_{0k}/\partial\mathbf{q}$ in the neighbourhood of the point $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ and thereby fulfil the role of slightly perturbing natural systems.

Theorem 18 [51]. Let $L(\mathbf{q}, \dot{\mathbf{q}})\in C^h(s_\varepsilon\times R^n)$ $(h>2)$ and let a natural number m, $2<m\leqslant h$ exist such that

$$L_0(\mathbf{q}) = L_{0m} + W$$

where L_{0m} is a form of degree m and the function W is of a higher order of smallness at the point $\mathbf{q}=\mathbf{0}$. Furthermore, let

$$\mathbf{f} = \mathbf{f}_{[s]} + \tilde{\mathbf{f}}$$

where $f_{[s]}=O(\|\mathbf{q}\|^s)$, $\tilde{\mathbf{f}}=o(\|\mathbf{q}\|^s)$ and the natural number s belongs to the interval $[(m+2)/2, h]$. The steady motion is then unstable if the form L_{0m} does not have a maximum in it.

The validity of the theorem is established using the proof of Theorem 4.

Since the estimate

$$\|\mathbf{f}\|^2 < \lambda\|\mathbf{q}\|^{m+2}, \quad \lambda = \text{const}$$

holds under the conditions of Theorem 18, $\|\mathbf{f}\|^2/\|\mathbf{q}\|^m=O(\|\mathbf{q}\|^2)$ and on the basis of condition 4 of the corollary to Theorem 16, we obtain $\|\mathbf{f}\|^2/\|\mathbf{q}\|^k=o(\|\mathbf{q}\|^\beta)$. Since the number $\alpha\in]0, 1[$, it can always be chosen in such a way that the exponent β satisfies the inequality $0<\beta<1$. Consequently, the corollary to Theorem 16 is an extension of Theorem 18.

Theorem 19 [34]. Let $L(\mathbf{q}, \dot{\mathbf{q}})\in C^1(D\times R^n)$ and, for a number $\varepsilon>0$ $(D\supset\bar{s}_\varepsilon)$ which may be as small as desired, let the following conditions be satisfied:

1. the function $L_0(\mathbf{q})$ can be represented in the form $L_0(\mathbf{q})=L_{0k}(\mathbf{q})+L_{0m}(\mathbf{q})+R(\mathbf{q})$, $R(\mathbf{q})=o(\|\mathbf{q}\|^m)$, where $L_{0k}(\mathbf{q})$, $L_{0m}(\mathbf{q})$ are, respectively, homogeneous functions of degrees $k>0$, $m>k$, $L_{0k}(\mathbf{q})\leqslant 0$;
2. the function W_m (the value of L_{0m} on the set of zeros of the function L_{0k}) does not have a maximum at the point $\mathbf{q}=\mathbf{0}$;
3. equality (3.2) is satisfied;
4. equality (3.3) is satisfied.

The steady motion is then unstable.

Corollary. If $L(\mathbf{q}, \dot{\mathbf{q}})\in C^{m+1}(D\times R^n)$, then condition 3 of the theorem can be omitted while condition 4 is replaced by the following

$$\lim_{\|\mathbf{q}\|\to 0}(\|\mathbf{f}\|^2/\|\mathbf{q}\|^m) = 0$$

Theorem 19 is proved by representing the system under investigation in the form

$$\dot{\mathbf{q}} = \partial H/\partial\mathbf{p}, \quad \dot{\mathbf{p}} = -\partial H/\partial\mathbf{q} + GA^{-1}\mathbf{p} \tag{3.5}$$

$$H = {}^1\!/\!{}_2(\|\mathbf{p}\|^2 + \mathbf{p}^TB(\mathbf{q})\mathbf{p}) - L_0(\mathbf{q}) = h = \text{const}, \quad B(\mathbf{0}) = 0$$

where $G=(g_{ij})=(\partial f_i/\partial q_j-\partial f_j/\partial q_i)$ is the matrix of the gyroscopic forces. Since, by the conditions of the theorem, the representation of Eqs (3.5) in the zeroth level set of the integral $H(\mathbf{q}, \mathbf{p})$ holds in the form

$$\dot{\mathbf{q}} = \partial H/\partial\mathbf{p}, \quad \dot{\mathbf{p}} = \partial(L_{0k} + L_{0m})/\partial\mathbf{q} + o(\|\mathbf{q}\|^{m-1+\alpha})$$

the scheme for the proof of Theorem 9 is applicable.

4. THE VARIATIONAL APPROACH TO THE QUESTION OF THE STABILITY OF CONSERVATIVE SYSTEMS

The use of Lyapunov methods lies at the foundation of the technique used to obtain the results on the inversion of the Lagrange–Dirichlet and Routh theorems which have been presented above. The structure of the right-hand sides of the corresponding differential equations and, in particular, the possibility of separating out the principal part of the potential of the forces, which is far from being successfully done in all cases, are important in such an approach. Other approaches to the investigation of stability have therefore been initiated. Hagedorn [44] turned to the description of the motion of a natural system using the variational equation in the form of the Jacobi principle

$$\delta \int_{\mathbf{q}^{(1)}}^{\mathbf{q}^{(2)}} \sqrt{h-\Pi(\mathbf{q})}\sqrt{d\mathbf{q}A(\mathbf{q})d\mathbf{q}} = 0$$

which enabled him to avoid a number of characteristic difficulties, associated with the separation of the principal part in the potential energy expression, which occur when Lyapunov methods are employed.

Theorem 20 [44]. Let $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^2(D\times R^n)$. The equilibrium position of a natural system is unstable if the potential energy has a strictly local maximum in it.

As a result of the interpretation of the problem of stability as a boundary-value problem, it has been shown [44] that, under the conditions of the theorem and for a constant energy integral $h>0$, a solution exists which passes through the point $\mathbf{q}=\mathbf{0}$ and amounts to a sphere $\|\mathbf{q}\|^2 = a = \text{const}$, where a is independent of h after a finite time.

Theorem 20 generalizes Lyapunov's theorem [19], in which the analytic case $(L(\mathbf{q}, \dot{\mathbf{q}}) \in C^\omega)$ was considered and the strictly local maximum in $\Pi(\mathbf{q})$ at the equilibrium position was determined by the set of the smallest sized terms in the expression for $\Pi(\mathbf{q})$.

As revealed later, Theorem 20 also remains true when there is not a strict maximum in $\Pi(\mathbf{q})$ at the equilibrium position and $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^1$ [17, 62, 46] and, also, in the case of a time-dependent Lagrangian L [4].

Finally, in [60], the requirement concerning the smoothness of the Lagrangian L is reduced to a minimum when the assertion about instability only has a meaning within the framework of the variational description of a real system which enables one to avoid differentiation by defining equilibrium as an extremum of $\Pi(\mathbf{q})$.

Theorem 21 [60]. Let $L(\mathbf{q}, \dot{\mathbf{q}}) \in C(D\times R^n)$. The position of equilibrium of a natural system is unstable if the potential energy has a local maximum in it.

Unlike in [44, 4, 17, 62], where classical methods of variational calculation and their modern modifications are used, the possibility of converting the initial problem on stability into a particular geometric problem involving the search for geodesics in spaces with an internal metric is important in the proof of Theorem 21. The formulation of the problem in such a form enables one to make use of Rinow's theorem on geodesics [57, p. 151] and to draw a conclusion, on the basis of this theorem, regarding the existence of a trajectory in $R^n_{\mathbf{q}}$ which joins the point $\mathbf{q}=\mathbf{0}$ with the boundary of a certain small sphere $\|\mathbf{q}\|^2 = \eta = \text{const}$, where the number η is independent of h, $h>0$.

Hagedorn's approach [44] made it possible to change to an investigation of the stability of steady motions by replacing system (1.1), (3.1) with a variational problem

$$\int_{\mathbf{q}^{(1)}}^{\mathbf{q}^{(2)}} \left[2\sqrt{h+L_0(\mathbf{q})}\sqrt{\frac{1}{2}\mathbf{q}'^T A(\mathbf{q})\mathbf{q}'} + \mathbf{f}^T(\mathbf{q})\mathbf{q}' \right] ds = \min, \quad \mathbf{q}' = d\mathbf{q}/ds$$

which represents the principle of least action in Jacobi form.

Theorem 22 [45]. Let $L(\mathbf{q},\ \dot{\mathbf{q}})\in C^2(D\times R^n)$. The steady motion is then unstable if the expression $L_0-\frac{1}{2}\mathbf{f}^TA^{-1}\mathbf{f}$ has a strictly local minimum in it.

The conclusion of Theorem 22 also remains valid in the case when the Lagrangian L is almost periodically dependent on the time t and $L_0-\frac{1}{2}\mathbf{f}^TA^{-1}\mathbf{f}\geqslant \mathbf{0}$ in the neighbourhood of steady motion [4] and, also, when $L(\mathbf{q},\ \dot{\mathbf{q}})\in C^1$ [17, 18].

Remark 1. Theorem 22 preceded the weaker elementary form in [44], in which the requirement was made that the function $L_1 dt$ should be a complete differential which is characteristic in the case of gyroscopic unconnected systems. An important stimulating role was apparently played by Karapetyan's paper [8] which is based on [44] in which gyroscopically connected systems were considered. Although the result in this paper is inferior in general to Theorem 22, it is nevertheless most likely that it stimulated Hagedorn to implement the idea, proposed in [44] and which is case in Theorem 22, in more complete form.

Since the Lagrangian L is determined with an accuracy up to $d/dt\psi(\mathbf{q})$, it is natural to exclude terms having the structure of $d/dt\psi(\mathbf{q})$ from the expression for $L(\mathbf{q},\ \dot{\mathbf{q}})$. In fact, these considerations lie at the foundation of the following result.

Theorem 23 [64]. Let $L(\mathbf{q},\ \dot{\mathbf{q}})\in C^2$ and let a function $\psi(\mathbf{q})\in C^3$ exist such that the following conditions are satisfied:

1. $\partial\psi/\partial\mathbf{q}\,|_{\mathbf{q}=0}=\mathbf{0}$;
2. $H(\mathbf{q},\ \mathbf{p})\,|_{\mathbf{p}=\partial\psi/\partial\mathbf{q}}<0\quad \forall\mathbf{q}\in s_\varepsilon\setminus\{0\}$,

where $H(\mathbf{q},\ \mathbf{p})$ is the Hamiltonian which corresponds to system (1.1), (3.1).

The steady motion is then unstable.

We note that the choice of the function $\psi(\mathbf{q})$ in each actual case is determined by the problem which is being considered since, by acting in a random manner, Hagedorn's condition may not only improve the situation but also aggravate it. If $\psi(\mathbf{q})\equiv 0$, we arrive at Theorem 22.

When account is taken of [17, 18], Theorem 23 remains valid if $L(\mathbf{q},\ \dot{\mathbf{q}})\in C^1$, $\psi(\mathbf{q})\in C^2$ and condition 2 is replaced by the less strict inequality $H(\mathbf{q},\ \partial\psi/\partial\mathbf{q})\leqslant 0\quad \forall\mathbf{q}\in s_\varepsilon\setminus\{0\}$.

Remark 2. In the case of the instability criteria which have been considered above, the quite rigorous requirement that there should be a local maximum in the function $\Pi(\mathbf{q})$ at the equilibrium position or a local minimum of the quantity $L_0(\mathbf{q})-\frac{1}{2}\mathbf{f}^TA^{-1}\mathbf{f}$ for steady motion is essential. In this case, the conditions that there should be a maximum and minimum respectively in the relevant quantities are not necessarily rigorous. This requirement can be considered as its own kind of analogue of the restrictions associated with the analytic structure of the Lagrangian which are customarily imposed when Lyapunov methods are employed.

In all fairness it should be mentioned that, although the constructive application of the variational principles of mechanics to problems of investigating the stability of an equilibrium, implemented in the papers by Hagedorn and his followers, refer to comparatively recent times, the origins of such applications go back further to Thomson and Tait [65] and Routh [59].

5. THE USE OF HAMILTON'S ACTION FUNCTION TO INVESTIGATE THE STABILITY OF CONSERVATIVE SYSTEMS

5.1. It is well known [1, 42, 54] that the initial equations (1.1) can be obtained from the condition of the stationarity of Hamilton's action

$$\delta S=\delta\int_0^{t_1} L(\mathbf{q},\dot{\mathbf{q}})d\tau=0$$

and, consequently, S may be interpreted as a carrier of information regarding the system under investigation. Allowing for this, in the expression for the action S we substitute a fixed value t_1 for the current t and we then consider S as a quantity, that is, an action function, which characterizes the motion with respect to the real trajectories of the system. Then, by noting that the Lagrangian L retains its sign during the motions of the system subject to certain additional restrictions and that $dS/dt = L$, we attempt to use the action function S in the analysis of stability, e.g. as an analogue of a Lyapunov function. Here, it is true that, on account of the integral character of the specification of the action function

$$S = \int_0^t L d\tau \tag{5.1}$$

the question arises as to the representation of S in a form which would be suitable for the investigation of stability.

In order to explain the difficulties which have arisen here, we will consider the well-known representation of the action function S in the form of Hamilton's principal function [42, 54].

Considering the case when $L = L(\mathbf{q}, \dot{\mathbf{q}}) \in C^2(D \times R^n)$, we substitute the quantities

$$\mathbf{q} = \mathbf{q}(t, \mathbf{q}_0, \dot{\mathbf{q}}_0), \quad \dot{\mathbf{q}} = \dot{\mathbf{q}}(t, \mathbf{q}_0, \dot{\mathbf{q}}_0) \tag{5.2}$$

$$\mathbf{q}_0 = \mathbf{q}(t = 0), \quad \dot{\mathbf{q}}_0 = \dot{\mathbf{q}}(t = 0)$$

into the integrand of the action function, having first replaced t in (5.2) by τ, where $\mathbf{q}(t, \mathbf{q}_0, \dot{\mathbf{q}}_0)$ is the general solution of system (1.1).

On integrating, we find that

$$S = \tilde{S}(\tau, \mathbf{q}_0, \dot{\mathbf{q}}_0)|_0^t \in C^{(1,1,1)}_{t, \mathbf{q}_0, \dot{\mathbf{q}}_0}(I \times s_\delta) \tag{5.3}$$

Here I ($I \subseteq R$) is the maximum integral for which the vector $(\mathbf{q}(t), \dot{\mathbf{q}}(t))$ belongs to the neighbourhood $s_\varepsilon = \{(\mathbf{q}, \dot{\mathbf{q}}) \in D \times R^n, \ \|\mathbf{q} \oplus \dot{\mathbf{q}}\| < \varepsilon\}$ subject to the condition that $(\mathbf{q}_0, \dot{\mathbf{q}}_0) \in s_\delta \subset s_\varepsilon$. Taking account of the fact that, below, one is concerned about the instability of an equilibrium position, we put $I = R$, with no loss in generality, since, in the case of finite I, there is a fortiori instability.

Let us suppose that the Jacobian $\partial \mathbf{q}/\partial \dot{\mathbf{q}}_0$ does not vanish. Then, on solving the first group of relationships (5.2) for $\dot{\mathbf{q}}_0$ and substituting the result obtained into (5.3), we arrive at the expression for the principal Hamiltonian function

$$S = \bar{S}(\tau, \mathbf{q}_0, \mathbf{q}(\tau))|_0^t \subset C^{(1,1,1)}_{t\mathbf{q}_0, \mathbf{q}}(I^* \times s_\eta) \tag{5.4}$$

where $I^* \subseteq I$ is the maximum interval in which the vector $\mathbf{q}$ belongs to the neighbourhood $s_\eta = \{\mathbf{q}, \ \mathbf{q} \in D, \ \|\mathbf{q} - \mathbf{q}_0\| < \eta\}$ of the point $\mathbf{q}_0$. Unfortunately, the existence of focal points, which are accompanied by the vanishing of the Jacobian $\partial \mathbf{q}/\partial \dot{\mathbf{q}}_0$, may act as an impediment to the representation of the action function S in the form of (5.4). Although the focal points have a zero Lebesgue measure with respect to D (a domain in R^n), in a significant number of cases they nevertheless do not enable one to determine the function $S(t, \mathbf{q}_0, \mathbf{q})$ in the whole neighbourhood of the equilibrium position under investigation.

Similar difficulties also arise in the case if, following Chetayev [41, p. 237], one uses the solutions of the Hamilton–Jacobi equation [5, 6] instead of the principal Hamiltonian function.

In order to avoid these difficulties, we solve relationships (5.2) for the whole set of initial values of $\mathbf{q}_0$ and $\dot{\mathbf{q}}_0$ which can always be done according to the initial assumptions with regard to L and I. In this connection, we note, in particular, that the Jacobian $\partial(\mathbf{q}, \mathbf{p})/\partial(\mathbf{q}_0, \mathbf{p}_0)$ is equal to unity in the Hamiltonian variables $(\mathbf{q}, \mathbf{p})$ which once again demonstrates the fundamental difference between this situation and the case which corresponds to obtaining the principal Hamiltonian function. As a result, allowing for the fact that L is independent of t, we obtain [20, p. 347; 54, p. 402] from (5.2) that

$$\mathbf{q}_0 = \mathbf{q}(-t, \mathbf{q}, \dot{\mathbf{q}}), \quad \dot{\mathbf{q}}_0 = \dot{\mathbf{q}}(-t, \mathbf{q}, \dot{\mathbf{q}}) \tag{5.5}$$

In connection with (5.5), also see [2, p. 104].

On substituting the quantities (5.5) into (5.3), we have

$$S = S^*(\tau, \mathbf{q}(\tau), \dot{\mathbf{q}}(\tau))|_0^t \in C^{(1,1,1)}_{t\mathbf{q}\dot{\mathbf{q}}}(I \times s_\varepsilon) \tag{5.6}$$

The action S in the form of (5.6), unlike the principal Hamiltonian function, is determined in the whole set of permissible motions in the neighbourhood of the equilibrium position under investigation when account is taken of the stipulations laid down above. Although the function S in representation (5.6) does not possess many important properties of the principal Hamiltonian function and, in particular, the property of the separability of the variable t and the phase variables in the conservative case, nevertheless, when investigating the stability, it is possible successfully to establish certain important properties of the function S which enable one to use it as an analogue of a Lyapunov function.

If the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ has critical points in the set of a non-zero level of the Jacobi (energy) integral, the action function $S(t, \mathbf{q}, \dot{\mathbf{q}})$, by monotonically increasing (decreasing) at these points when $t \to \infty$ $(-\infty)$ (or on phase trajectories which are attracted to them) reaches, as fixed points of the phase space, its own limiting values of $\pm\infty$ when $t \in \overline{R} = R \cup \{-\infty, \infty\}$. Below, we shall therefore confine ourselves to the treatment of a class of systems for which $\operatorname{grad} L(\mathbf{q}, \dot{\mathbf{q}})$ does not vanish, at least, in the set of negative values of the Jacobi (energy) integral in order to avoid complications associated with these singularities of the function S.

Passing directly to the criteria for the instability of an equilibrium, obtained using Hamilton's action function, we shall dwell on the most fundamental criteria which are of the greatest interest, including criteria which are useful in applications. Here, we shall take greater trouble with the conceptual aspect of the question without pursuing the aim of listing all those results appertaining to this set.

Theorem 24 [30]. For an $\varepsilon > 0$ $(D \supset s_\varepsilon^* = \{\mathbf{q} \in R^n, \ \|\mathbf{q}\| \leq \varepsilon\})$ which may be as small as desired, let the following conditions be satisfied:

1. $\omega = \{\mathbf{q} \in s_\varepsilon^* : \ \Pi(\mathbf{q}) < 0\} \neq \phi$;
2. $0 \in \partial\omega$;
3. $\partial\Pi/\partial\mathbf{q} \neq \mathbf{0} \quad \forall \mathbf{q} \in s_\varepsilon^* \backslash \partial\omega$.

The equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of a natural system is then unstable.

The proof of the theorem rests on the following lemma.

Lemma 2. Subject to the conditions of Theorem 24, the action function S permits a representation in which the corresponding "primitive" function $S^*(\tau, \mathbf{q}, \dot{\mathbf{q}})$ satisfies the relationship

$$S^*(\tau, \mathbf{q}, -\dot{\mathbf{q}}) = -S^*(\tau, \mathbf{q}, \dot{\mathbf{q}}) \forall (\mathbf{q}, \dot{\mathbf{q}}) \subset \Omega \tag{5.7}$$

on the trajectories of system (1.1), (1.2) belonging to the set $\Omega = \{(\mathbf{q}, \ \dot{\mathbf{q}}) \in s_\varepsilon : \ T + \Pi = h = 0\}$.

Proof. Representing the initial natural system in the form of two equivalent Hamiltonian systems

$$\dot{\mathbf{q}} = \partial H_i / \partial \mathbf{p}, \quad \dot{\mathbf{p}} = -\partial H_i / \partial q, \quad i = 1,2 \tag{5.8}$$

$$H_i = (-1)^{i+1}(^1/_2\mathbf{p}^T A^{-1}\mathbf{p} + \Pi) = (-1)^{i+1}h, \quad h = \text{const} \tag{5.9}$$

where, for convenience, we have used the same notation for the dependent variables, let us consider the Hamiltonian action functions corresponding to these systems

$$S_i = \int\limits_0^t (\mathbf{p}\dot{\mathbf{q}} - H_i)d\tau, \quad i = 1,2 \tag{5.10}$$

Since systems (5.8), when $i = 1, 2$, respectively, pass into one another on replacing p by $(-p)$ and the inequalities

$$\mathbf{p}\,\dot{\mathbf{q}} - H_i = (-1)^{i+1}(^1/_2\mathbf{p}^T A^{-1}\mathbf{p} - \Pi)$$

hold by virtue of (5.8) and (5.9), the functions S_i, according to (5.10), differ solely in their signs, that is, $S_2 = -S_1$.

Let us now consider the set $\tilde{\Omega} = \Omega_j = \{(\mathbf{q}, \ \mathbf{p}) \in \tilde{s}_\varepsilon : \ H_i = 0\}$, representing in it the equalities (5.10) in the form

$$S_i = \int_0^t \mathbf{p}d\mathbf{q} \quad \forall(\mathbf{q},\mathbf{p}) \subset \tilde{\Omega} \tag{5.11}$$

The relationship $S_2 = -S_1$ in $\tilde{\Omega}$ can then be interpreted as the result of the replacement in the integrand of equality (5.11) of $\mathbf{p}$ by $(-\mathbf{p})$. Since the replacement of $\mathbf{p}$ by $(-\mathbf{p})$, using Eqs (5.8), is equivalent to the replacement of $\dot{\mathbf{q}}$ by $(-\dot{\mathbf{q}})$ and systems (5.8) are equivalent to (1.1), (1.2) then, consequently, the function $S^*(\tau, \mathbf{q}, \dot{\mathbf{q}})$, like the accompanying initial system (1.1), (1.2), satisfies equality (5.7) in the set Ω.

The existence of a class of positive semitrajectories Γ^+ passing through Ω, on which the function S is bounded, at least, within the limits of the neighbourhood s_ε being considered follows from (5.7) when account is taken of the reversibility of a natural system. This enables one to use S as an analogue of a Lyapunov function (precisely an analogue since S does not satisfy any classical Lyapunov theorem or its modification).

Corollary. Let the Lagrangian L in the neighbourhood of the point $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ be analytic with respect to $\mathbf{q}$. The equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of a natural system is then unstable if the potential energy $\Pi(\mathbf{q})$ at the point $\mathbf{q} = \mathbf{0}$ does not have a local minimum.

Actually, in this case for sufficiently small $\varepsilon > 0$, all the critical points of the function $\Pi(\mathbf{q})$ belong to the set $\partial\omega \cap s_\varepsilon^*$ [61] and, consequently, the conditions of Theorem 24 are satisfied.

Remark 1. Its particular version [27] preceded Theorem 24. In this version $\partial\omega$ is replaced by $\{0\}$ in condition 3. It is then additionally possible to prove the existence of motions which are asymptotic to the equilibrium position under investigation and, thereby, the rough instability of the equilibrium [14, p. 102] (in this connection, see also [28]).

Remark 2. At first glance, it may appear strange that the function S^*, according to Lemma 2, is of variable sign even in the case of the sign determinacy of the corresponding Lagrangian in the neighbourhood of the equilibrium position. In order to show that this is actually so and as an illustration of the lemma, let us take the example of a natural system with a single degree of freedom, the Lagrangian of which has the form

$$L = \tfrac{1}{2}(\dot{q}^2 + q^2) \tag{5.12}$$

Representing the general solution of the corresponding equation in the form

$$q = C_1 e^t + C_2 e^{-t} \tag{5.13}$$

let us differentiate the function q with respect to t and substitute the expressions for q and $\dot{q}$ into (5.12). On carrying out the integration in (5.1) using expressions (5.12) and (5.13), we find that

$$S = \tfrac{1}{2}(C_1^2 e^{2\tau} - C_2^2 e^{-2\tau})\big|_0^t$$

Finally, on eliminating the constants C_1 and C_2 using the expressions for q and $\dot{q}$ we obtain

$$S = \tfrac{1}{2} q\dot{q}\big|_0^t \tag{5.14}$$

which agrees with Lemma 2.

Furthermore, we also arrive at an analogous expression in the case of linear systems of the form of (1.1) with n degrees of freedom in which the identity

$$L = \dot{\mathbf{q}}\,\mathbf{p} - H = (\mathbf{q}\mathbf{p})^{\cdot} + \mathbf{q}\partial H/\partial \mathbf{q} - H = (\mathbf{q}\mathbf{p})^{\cdot} - L, \quad \mathbf{p} = \partial L/\partial \dot{\mathbf{q}}$$

is readily proved by direct verification.

We note that Lyapunov [19] further used the function $V = \mathbf{q}\mathbf{p}$, which, according to what has been stated above, is a doubled Hamiltonian action function apart from a constant in the case of linear conservative systems, to investigate the stability of equilibrium.

5.2. The Hamiltonian action function can also be used to investigate the stability of stationary motions (the equilibrium position $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ of system (1.1), (3.1) [29, 31]. Although the corresponding equations of the perturbed motion (1.1), (3.1) are not invertible ($L(\mathbf{q},\ \dot{\mathbf{q}})\neq L(\mathbf{q},\ -\dot{\mathbf{q}})$), this does not introduce any fundamentally new points into the technique of the investigation. In this case, it is simply advisable in the technical plan to consider the Lagrangian $L(\mathbf{q},\ -\dot{\mathbf{q}})$ together with the Lagrangian $L(\mathbf{q},\ \dot{\mathbf{q}})$.

Instead of implementing this idea, let us proceed in a somewhat different manner and construct an analogue of the Lyapunov function in a form such that the action function S contains in it, as one of the variables, the question of the boundedness (unboundedness) which is still not determined. A non-trivial example of the use of a sign-variable auxiliary function with a sign-variable derivative, which nevertheless enables one to analyse the stability of equilibrium completely in the spirit of the idea of the second Lyapunov method, can be successfully demonstrated in this way.

Theorem 25 [37]. For an $\varepsilon>0$ $(D\supset\bar{s}_\varepsilon^*)$ which may be as small as desired, let the following conditions be satisfied

1. $\omega=\{\mathbf{q}\in s_\varepsilon^*:\ L_0(\mathbf{q})>0\}\neq\phi$;
2. $0\in\partial\omega$;
3. $\partial L_0/\partial\mathbf{q}\neq\mathbf{0}\ \ \forall\mathbf{q}\in\omega$;
4. $L_0-\frac{1}{2}\mathbf{f}^TA^{-1}\mathbf{f}\geqslant 0\ \ \forall\mathbf{q}\in\omega$.

The steady motion (the equilibrium position $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ of system (1.1), (3.1)) is then unstable.

The proof is based on the representation of system (1.1), (3.1) in Hamiltonian form and the treatment of the auxiliary function $V=\mathbf{qp}(S^2+1)^{-1}$ in the set $\Omega^-=\{(\mathbf{q},\ \mathbf{p})\in s_\varepsilon:\ H=h<0\}$ which, according to condition 3 of Theorem 25, is the manifold for any fixed $h>0$ of sufficiently small modulus.

On calculating the derivative along the vector field of the system under consideration, we have

$$\frac{dV}{dt}=\frac{L}{S^2+1}(1-\lambda)-\frac{\mathbf{q}\partial H/\partial\mathbf{q}-H}{S^2+1},\quad \lambda=2\mathbf{qp}\frac{S}{S^2+1} \tag{5.15}$$

and, consequently, the derivative, except in certain special cases, is sign variable in the set Ω^-.

Assuming the stability of the equilibrium position and, thereby, of the re-entry of almost all of the trajectories belonging to Ω, according to Poincaré's theorem [54, p. 439], we integrate (5.15) along the trajectories γ_1^*, which possess the property of re-entry. As a result, we obtain the equality

$$\left.\frac{\mathbf{qp}}{S^2+1}\right|_{t_k}^{t_k+\sigma(k)}=\operatorname{arctg}S|_{t_k}^{t_k+\sigma(k)}+o(\operatorname{arctg}S|_{t_k}^{t_k+\sigma(k)})+\int_{t_k}^{t_k+\sigma(k)}\frac{H-\mathbf{q}\partial H/\partial\mathbf{q}}{S^2+1}dt \tag{5.16}$$

Here, $\sigma(k)$ denotes a sufficiently small positive number such that

$$\gamma_1^+|_{t_k}^{t_k+\sigma(k)}\subset\Omega_1^-=\{(\mathbf{q},\mathbf{p})\in\Omega^-:\ H-\mathbf{q}\partial H/\partial\mathbf{q}>0\} \tag{5.17}$$

Under the assumptions of Theorem 25, it can be shown that $\Omega_1^-\neq\phi$ and, using (5.16) and (5.17), we arrive at a contradiction since, when $S\to\infty$, $\mathbf{qp}(S^2+1)^{-1}=o(\operatorname{arctg}S)$.

Corollary. Let a system be natural $(L_1\equiv 0)$ and, for an $\varepsilon>0$ $(D\supset\bar{s}_\varepsilon^*)$ which may be a small as desired, let the following conditions be satisfied

1. $\omega=\{\mathbf{q}\in s_\varepsilon^*:\ \Pi(\mathbf{q})\}\neq\phi$;
2. $0\in\partial\omega$;
3. $\partial\Pi/\partial\mathbf{q}\neq 0\ \ \forall\mathbf{q}\in\omega$.

The equilibrium position $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ of system (1.1), (1.3) is then unstable.

Remark 1. If one assumes the existence of a connected component $\omega^*\subset\omega$ for which $0\in\partial\omega^*$, then, under the conditions 3 and 4 of Theorem 25, ω can be replaced by ω^*.

Remark 2. If the expression $L_0-\frac{1}{2}\mathbf{f}^TA^{-1}\mathbf{f}$ has a local (not necessarily strict) minimum at the point $\mathbf{q}=\mathbf{0}$, then, by considering the motion of the system in the set

$$\Omega^+=\{(\mathbf{q},\ \mathbf{p})\in s_\varepsilon:\ H=h>0\}$$

it is possible to obtain an extension of Hagedorn's result [45], analogous to that obtained in [4]. Actually, in this case, the set Ω^+ is a manifold for any fixed sufficiently small $h>0$ and, without condition 3 of Theorem 25, conditions 1 and 2 are automatically satisfied.

The action function S can also be successfully used to investigate the stability of non-holonomic Chaplygin systems [33].

Finally, we note that there is a certain similarity between the theorems obtained using variational principles and Theorems 24 and 25 of Section 5 since, in each of these cases, the conditions of the theorem assume that the functions $\Pi(\mathbf{q})$, $L_0(\mathbf{q})$ or $L_0(\mathbf{q})-\frac{1}{2}\mathbf{f}^T A^{-1}\mathbf{f}$ have a certain quality but make no assumption regarding their analytic structure which is associated with the separation of the principal part, just as was the case in Sections 1–3.

6. CONCLUSION

Summing up the results on the inversion of the Lagrange–Dirichlet and Routh theorems which have been presented above, we can say that the Lyapunov methods remain one of the most powerful means of investigating the stability of conservative systems. At the same time, the concepts incorporated in them allow of further refinement and development depending on the problems which these methods are used to solve. The investigations in [11–13] which, together with [15, 16] extend the possibilities of the first Lyapunov method, are significant in this connection.

As far as the inversion theorems obtained by the use of the second Lyapunov method and, especially, Chetayev's theorem on instability, are concerned, here the specific details of the systems under consideration and, in particular, their conservativeness were made use of, whenever possible, when constructing auxiliary functions. On the basis of the latter, it was found to be convenient to consider the set of the zeroth level of the energy integral or the Jacobi integral, when there is no minimum in the potential energy at the equilibrium position as a set possessing the properties of an absolute sector. This fact introduces a simplification into the construction of the Lyapunov–Chetayev function, the necessary properties of which can only be reasonably attributed to a set with a Lebesgue measure equal to zero.

Finally, the use of the Hamiltonian action function, which depends on time and the phase coordinates, to investigate the stability of conservative systems, with all the specific details implied in such an approach, may be considered as a useful step in the direction of making use of new possibilities incorporated in the second Lyapunov method.

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